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Field-Theoretic Approach to Critical Dynamics Far from Equilibrium

Tomoji Yamada,[1] Takao Ohta,[2] and Kyozi Kawasaki[2]

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The field-theoretic approach to critical phenomena is extended to deal with critical dynamics far from equilibrium. In particular, the macroscopic evolution equation for the average order parameter is derived in a manner parallel to the derivation of the equation of state. The method is illustrated by deriving the scaled macroscopic equation of motion for the time-dependent Ginzburg–Landau model near the critical point for dimensionality near four.

KEY WORDS: Critical phenomena; far from equilibrium; time-dependent Ginzburg–Landau model.

1. INTRODUCTION

Recently, the method of renormalized field theory successful in equilibrium critical phenomena(1) was extended to dynamic critical phenomena.(2–4) It was realized, furthermore, that such an approach is also useful to study phenomena near a critical point, where deviations from equilibrium are not necessarily small.(5) Here we further develop the approach by presenting a method for calculating the generating functional Γ [Eq. (45)], which plays the central role in this approach,(4) and the method is illustrated by deriving from Γ the macroscopic evolution equation for the spatially uniform average order parameter for the time-dependent Ginzburg–Landau (TDGL) model, which conforms to the dynamic scaling law.(6)

In the following section we present the general perturbation scheme for obtaining Γ; the TDGL model is discussed in Section 4.

We will see that our treatment of critical dynamics far from equilibrium closely parallels that of equilibrium critical phenomena. In particular, our

[1] Department of Applied Science, Kyushu University, Fukuoka, Japan.
[2] Department of Physics, Kyushu University, Fukuoka, Japan.

derivation of the macroscopic evolution equation for the average order parameter parallels that of the equation of state.

2. FIELD-THEORETIC METHOD

Let us start with the general stochastic model equation for the probability distribution function $P(\mathbf{S}, t)$ of the variables $\mathbf{S} = \{S_1, S_2, ...\}$ in vector notation:

$$[(\partial/\partial t) + H(\tilde{\mathbf{S}}, \mathbf{S})]P(\mathbf{S}, t) = 0 \tag{1}$$

where

$$\tilde{S}_i \equiv -\partial/\partial S_i \tag{2}$$

and the index i or S specifies not only the type of variable but also the quasi-continuous wave vector or spatial coordinate. In the stochastic operator H, $\tilde{S}_i$ always comes to the right of all the S's.

We now introduce the time-dependent source fields $\mathbf{h}^t = \{h_1{}^t, h_2{}^t, ...\}$ and $\tilde{\mathbf{h}}^t = \{\tilde{h}_1{}^t, \tilde{h}_2{}^t, ...\}$ as in Refs. 4 and 5 and define the total stochastic operator H^t by

$$H^t(\tilde{\mathbf{S}}, \mathbf{S}) = H(\tilde{\mathbf{S}}, \mathbf{S}) - \tilde{\mathbf{h}}^t \cdot \tilde{\mathbf{S}} - \mathbf{h}^t \cdot \mathbf{S} \tag{3}$$

The stochastic equation

$$[(\partial/\partial t) + H^t(\tilde{\mathbf{S}}, \mathbf{S})]P^t(\mathbf{S}) = 0 \tag{4}$$

describes the time evolution of $P^t(\mathbf{S})$, which is in general not normalized in the presence of $\mathbf{h}^t$. We now write

$$\tilde{\mathbf{S}} = \tilde{\mathbf{m}}^t + \tilde{\boldsymbol{\phi}}^t \tag{5a}$$

$$\mathbf{S} = \mathbf{m}^t + \boldsymbol{\phi}^t \tag{5b}$$

where $\tilde{\mathbf{m}}^t$ and $\mathbf{m}^t$ are some functions of space and time to be specified later. Substituting (5) into H^t, which is then expanded in $\tilde{\boldsymbol{\phi}}^t$ and $\boldsymbol{\phi}^t$, we find that H takes the form

$$H(\tilde{\mathbf{S}}, \mathbf{S}) = \sum_{ln} \frac{1}{l!\,n!} \sum_{\{i\}} \sum_{\{j\}} J_{ln}^t(\{i\}\{j\})\tilde{\phi}_{i_1}^t \cdots \tilde{\phi}_{i_l}^t \phi_{j_1}^t \cdots \phi_{j_n}^t \tag{6}$$

where J_{ln}^t is written more explicitly as $J_{ln}(\{i\}, \{j\}, \tilde{\mathbf{m}}^t, \mathbf{m}^t)$. In the following it is convenient to introduce the reference distribution function $P_0{}^t(\mathbf{S})$ given by

$$P_0{}^t(\mathbf{S}) = \delta(\mathbf{S} - \mathbf{m}^t) \tag{7}$$

with $\delta(\mathbf{x}) \equiv \prod_i \delta(x_i)$; we then readily find

$$[(\partial/\partial t) + H_0{}^t]P_0{}^t(\mathbf{S}) = 0 \tag{8}$$

with

$$H_0{}^t = -\dot{\mathbf{m}}^t \cdot \tilde{\boldsymbol{\phi}}^t + \dot{\tilde{\mathbf{m}}}^t \cdot \boldsymbol{\phi}^t - \dot{\mathbf{m}}^t \cdot \tilde{\mathbf{m}}^t + \sum_{ij} J_{11}^t(ij)\tilde{\phi}_i{}^t\phi_j{}^t \tag{9}$$

We now split H^t as

$$H^t(\tilde{\mathbf{S}}\mathbf{S}) = C^t + H_0{}^t + H_{\mathrm{I}}{}^t \tag{10}$$

$$H_{\mathrm{I}}^t \equiv H_1{}^t + H_2{}^t \tag{11}$$

where

$$C^t = C(\tilde{\mathbf{m}}^t, \mathbf{m}^t, \tilde{\mathbf{h}}^t, \mathbf{h}^t) \equiv H(\tilde{\mathbf{m}}^t, \mathbf{m}^t) + \tilde{\mathbf{m}}^t \cdot \dot{\mathbf{m}}^t - (\tilde{\mathbf{h}}^t \cdot \tilde{\mathbf{m}}^t + \mathbf{h}^t \cdot \mathbf{m}^t) \tag{12}$$

$$H_1{}^t \equiv \mathbf{M}^t \cdot \tilde{\boldsymbol{\phi}}^t + \tilde{\mathbf{M}}^t \cdot \boldsymbol{\phi}^t \tag{13}$$

with

$$\mathbf{M}^t \equiv \mathbf{J}_{10}^t + \dot{\mathbf{m}}^t - \tilde{\mathbf{h}}^t, \qquad \tilde{\mathbf{M}}^t = \mathbf{J}_{01}^t - \dot{\tilde{\mathbf{m}}}^t - \mathbf{h}^t \tag{14}$$

where $\mathbf{J}_{10}^t$ and $\mathbf{J}_{01}^t$ are the vectors whose ith components are $J_{10}^t(i)$ and $J_{01}^t(i)$, respectively. Finally, H_2 is the remainder, which is identical to $H(\tilde{\mathbf{S}}, \mathbf{S})$ except that $J_{00}^t, J_{10}^t, J_{01}^t$, and J_{11}^t are set to zero.

Let us now suppose that $P^t(\mathbf{S})$ coincides with $P_0{}^t(\mathbf{S})$ at some distant time in the past $t = t_0$. This should not be a real restriction for ergodic systems since the memory of the initial distribution will be lost beyond a certain time after t_0, which can be chosen to be arbitrarily far into the past. The formal solution of (4)

$$P^t(\mathbf{S}) = \left\{\exp_+\left[-\int_{t_0}^t ds\, H^s(\tilde{\mathbf{S}}, \mathbf{S})\right]\right\}P_0^{t_0}(\mathbf{S}) \tag{15}$$

where $\exp_+$ is the usual time-ordered exponential, can be rewritten for $t = t_f$ as

$$\begin{aligned} P^{t_f}(\mathbf{S}) &= \left\{\exp\left(-\int_{t_0}^{t_f} dt\, C^t\right) \exp_+\left[-\int_{t_0}^{t_f} H_{\mathrm{I}}(t)\right] \exp_+\left(-\int_{t_0}^{t_f} dt\, H_0{}^t\right)\right\}P_0^{t_0}(\mathbf{S}) \\ &= \left\{\exp\left(-\int_{t_0}^{t_f} dt\, C^t\right) \exp_+\left[-\int_{t_0}^{t_f} dt\, H_{\mathrm{I}}(t)\right]\right\}P_0^{t_f}(\mathbf{S}) \end{aligned} \tag{16}$$

where

$$H_{\mathrm{I}}(t) \equiv \left[\exp_+\left(-\int_t^{t_f} ds\, H_0{}^s\right)\right]H_{\mathrm{I}}{}^t \exp_-\left(\int_t^{t_f} ds\, H_0{}^s\right) \tag{17}$$

Here t_f is some distant time in the future where we assume $\tilde{\mathbf{m}}^t$ to vanish:

$$\tilde{\mathbf{m}}^{t_f} = 0 \tag{18}$$

Introducing the notation

$$\int dS \cdots = \langle 0| \cdots \tag{19}$$

we obviously have

$$\langle 0|\tilde{\boldsymbol{\phi}}^{t_f} = 0 \tag{20}$$

Now, the reference distribution function (7) is not a good first approximation for many systems of physical interest and we construct instead a Gaussian distribution function P_g^t, which obeys

$$[(\partial/\partial t) + H_g^t]P_g^t = 0 \tag{21}$$

with

$$H_g^t = H_0^t + \tfrac{1}{2}\sum_{ij} J_{20}^t(ij)\tilde{\phi}_i^t\tilde{\phi}_j^t \tag{22}$$

and with the initial condition $P_g^{t_0} = P_0^{t_0}$. We then introduce the ket vector $|0\rangle$ by

$$|0\rangle \equiv P_g^{t_f} \tag{23}$$

and then we find

$$P^{t_f}(\mathrm{S}) = \left\{\exp\left(-\int_{t_0}^{t_f} dt\, C^t\right) \exp_+\left[-\int_{t_0}^{t_f} dt\, H'(t)\right]\right\}|0\rangle \tag{24}$$

$$H'(t) \equiv \left[\exp_+\left(-\int_t^{t_f} ds\, H_g^s\right)\right]H'^t \exp_-\left(\int_t^{t_f} ds\, H_g^s\right) \tag{25}$$

where H'^t is identical to H_{I}^t except that J_{20}^t is set to zero. $H'(t)$ is obtained from H'^t by replacing ϕ^t and $\tilde{\phi}^t$ by $\phi(t)$ and $\tilde{\phi}(t)$, defined by

$$\begin{bmatrix}\boldsymbol{\phi}(t)\\ \tilde{\boldsymbol{\phi}}(t)\end{bmatrix} = \left[\exp_+\left(-\int_t^{t_f} H_g^s\, ds\right]\right)\begin{bmatrix}\boldsymbol{\phi}^t\\ \tilde{\boldsymbol{\phi}}^t\end{bmatrix} \exp_-\left(\int_t^{t_f} H_g^s\, ds\right) \tag{26}$$

Since the commutators of H_g^t and ϕ_i^t and $\tilde{\phi}_i^t$ are linear combinations of ϕ_j^t and $\tilde{\phi}_j^t$, (26) can be written as a transformation between the two vectors,

$$\Phi(t) \equiv \begin{bmatrix}\boldsymbol{\phi}(t)\\ \tilde{\boldsymbol{\phi}}(t)\end{bmatrix}, \qquad \Phi \equiv \begin{bmatrix}\boldsymbol{\phi}^{t_f}\\ \tilde{\boldsymbol{\phi}}^{t_f}\end{bmatrix} \tag{27}$$

in the form

$$\Phi(t) = \left[\exp_-\left(\int_t^{t_f} ds\, \mathscr{L}^s\right)\right]\Phi \tag{28}$$

where

$$\mathscr{L}^t = \begin{bmatrix} J_{11}^t & J_{20}^t \\ 0 & -(J_{11}^t)^T \end{bmatrix} \tag{29}$$

and J_{11}^t and J_{20}^t are the matrices whose ij elements are $J_{11}^t(ij)$ and $J_{20}^t(ij)$, respectively, and the superscript T denotes the transpose of a matrix. On the other hand, by (21) we can verify that

$$\begin{aligned} P_g^{t_f} &= \left[\exp_+\left(-\int_{t_0}^{t_f} dt\, H_g^{\,t}\right)\right] P_0^{t_0} \\ &= \left\{\exp_+\left[-\int_{t_0}^{t_f} dt\, \tfrac{1}{2}\sum_{ij} J_{20}^t(ij)\tilde{\psi}_i(t)\tilde{\psi}_j(t)\right]\right\} P_0^{t_f} \end{aligned} \tag{30}$$

where

$$\begin{aligned} \tilde{\boldsymbol{\psi}}(t) &= \left[\exp_+\left(-\int_t^{t_f} ds\, H_0^{\,s}\right)\right]\tilde{\boldsymbol{\phi}}^t \exp_-\left(\int_t^{t_f} ds\, H_0^{\,s}\right) \\ &= \left\{\exp_-\left[-\int_t^{t_f} ds\, (J_{11}^s)^T\right]\right\}\cdot\tilde{\boldsymbol{\phi}}^{t_f} \end{aligned} \tag{31}$$

This means that, by (19) and (20), $P_g^{t_f}$ is properly normalized since $P_0^{t_f}$ is normalized:

$$\int d\mathbf{S}\, P_g^{t_f}(\mathbf{S}) = \langle 0|0\rangle = 1 \tag{32}$$

Using (26) and (30), we obtain

$$\boldsymbol{\phi}(t_0)|0\rangle = \left[\exp_+\left(-\int_{t_0}^{t_f} ds\, H_g^{\,s}\right)\right]\boldsymbol{\phi}^{t_0}P_0^{t_0} = 0 \tag{33a}$$

Also, since by (28) and (29), $\tilde{\phi}_i(t)$ is a linear combination of $\tilde{\phi}_j^{t_f}$, we have by (20)

$$\langle 0|\tilde{\boldsymbol{\phi}}(t_0) = 0 \tag{33b}$$

At this point the analogy of our formalism with that of the quantum field theory is evident. $\phi_i(t_0)$ and $\tilde{\phi}_i(t_0)$ are annihilation and creation operators of a particle in the state i, respectively, and the bra and ket vectors $\langle 0|$ and $|0\rangle$ represent the free vacuum state. The time evolution of the free vacuum state in the interaction representation in the presence of external source fields $\mathbf{h}^t$ and $\tilde{\mathbf{h}}^t$ is described by the S-matrix operator [see (24)]:

$$S(t_f, t_0) \equiv \exp_+\left[-\int_{t_0}^{t_f} dt\, H'(t)\right] \tag{34}$$

The applicability of Wick's theorem for expectation values like

$$\langle 0|\Phi_i(t_i)\Phi_j(t_j)\cdots\Phi_l(t_l)|0\rangle \tag{35}$$

is evident, where, however, two types of free propagators are required because $H_g{}^t$ contains processes in which two particles are created from the vacuum:

$$G_{ij}(tt') = \langle 0|(\phi_i(t)\tilde{\phi}_j(t'))_+|0\rangle \tag{36}$$

$$F_{ij}(tt') = \langle 0|(\phi_i(t)\phi_j(t'))_+|0\rangle \tag{37}$$

where $(\cdots)_+$ denotes time ordering. We also have

$$\langle 0|(\tilde{\phi}_i(t)\tilde{\phi}_j(t'))_+|0\rangle = 0 \tag{38}$$

These propagators satisfy the following equations:

$$\frac{\partial}{\partial t} G_{ij}(tt') = -\sum_k (J_{11}^t)_{ik} G_{kj}(tt') + \delta_{ij}\,\delta(t-t') \tag{39}$$

$$\frac{\partial}{\partial t} F_{ij}(tt') = -\sum_k (J_{11}^t)_{ik} F_{kj}(tt') - \sum_k (J_{20}^t)_{ik} G_{jk}(t't) \tag{40}$$

which are to be solved under the conditions

$$G_{ij}(tt') = 0 \qquad \text{for} \quad t < t'$$
$$F_{ij}(tt) = \sigma_{ij}^g(t)$$

where $\sigma_{ij}^g(t)$ is the variance of $\phi_i(t)$ for the Gaussian distribution $P_g^{t_f}$.

So far $\tilde{\mathbf{m}}^t$ and $\mathbf{m}^t$ are completely arbitrary except that $\tilde{\mathbf{m}}^{t_f}$ should vanish. In fact, the most convenient choice for them is to determine them self-consistently so that $\tilde{\mathbf{m}}^t$ and $\mathbf{m}^t$ become the averages of $\tilde{\mathbf{S}}^t$ and $\mathbf{S}^t$ in the presence of source fields, respectively. This will be done in the next section using the generating functionals.

Feynman diagrams for the perturbation series of (34) can be easily constructed, where we let the time run from the right to the left. See Fig. 1, which is self-explanatory.

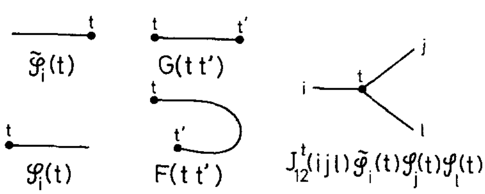

Fig. 1. Some elements of Feynman diagrams.

3. GENERATING FUNCTIONALS

Renormalized field theory for critical dynamics far from equilibrium is conveniently couched in the language of generating functionals.[4,5] First we consider

$$W\{\tilde{\mathbf{h}}, \mathbf{h}\} \equiv \ln \int d\mathbf{S}\, P^{t_f}(\mathbf{S})$$
$$= -\int_{t_0}^{t_f} dt\, C(\tilde{\mathbf{m}}^t, \mathbf{m}^t, \tilde{\mathbf{h}}^t, \mathbf{h}^t) + \ln\langle 0|S(t_f, t_0)|0\rangle \tag{41}$$

where two terms on the right-hand side depend on $\tilde{\mathbf{m}}^t$ and $\mathbf{m}^t$ as well. Such dependences, however, should cancel each other because the original definition of W does not contain $\{\tilde{\mathbf{m}}^t, \mathbf{m}^t\}$. We now choose $\tilde{\mathbf{m}}^t$ and $\mathbf{m}^t$ by

$$\tilde{\mathbf{m}}^t = \delta W/\delta \tilde{\mathbf{h}}^t \tag{42a}$$

$$\mathbf{m}^t = \delta W/\delta \mathbf{h}^t \tag{42b}$$

This implies, in view of (12) and (41), that

$$\frac{\delta}{\delta \tilde{\mathbf{h}}^t} \ln\langle 0|S|0\rangle = \frac{\delta}{\delta \mathbf{h}^t} \ln\langle 0|S|0\rangle = 0 \tag{43}$$

or

$$\langle 0|(\tilde{\phi}(t)S)_+|0\rangle/\langle 0|S|0\rangle = \langle 0|(\phi(t)S)_+|0\rangle/\langle 0|S|0\rangle = 0 \tag{44}$$

where $\tilde{\mathbf{m}}^t$ and $\mathbf{m}^t$ are kept fixed during the differentiations in (43).

Diagrammatically, the second term of (41) is represented by connected graphs without external lines such as those shown in Fig. 2. The conditions (44) are shown in Fig. 3, where cross-hatched circles represent the sum of all the connected graphs with one leg.

The generating functional $W\{\tilde{\mathbf{h}}, \mathbf{h}\}$ produces through differentiations with respect to $\tilde{\mathbf{h}}^t$ and $\mathbf{h}^t$ the various response and correlation functions.[4] These

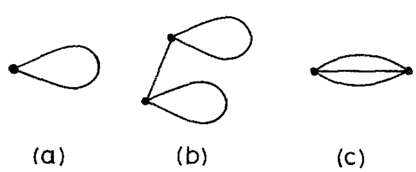

Fig. 2. Diagrams contributing to W.

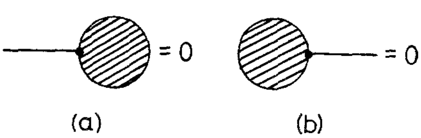

Fig. 3. Diagrams representing the conditions (44).

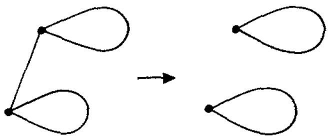

Fig. 4. Severing of articulation lines.

functions in general contain long range space–time correlations, which arise in graphical terms from articulation lines, the removal of which severs a connected graph into more than two disconnected parts, as is exemplified in Fig. 4. Direct calculation of $W\{\tilde{\mathbf{h}}, \mathbf{h}\}$ in a reasonable approximation hence is rather cumbersome, requiring various sorts of resummations of graphs, which in effect amounts to solving "hydrodynamic" equations. A better alternative is to consider the following Legendre transformation:[1,4,5]

$$\Gamma\{\tilde{\mathbf{m}}, \mathbf{m}\} = -W\{\tilde{\mathbf{h}}, \mathbf{h}\} + \int dt\, (\tilde{\mathbf{h}}^t \cdot \tilde{\mathbf{m}}^t + \mathbf{h}^t \cdot \mathbf{m}^t)$$
$$= \int [H(\tilde{\mathbf{m}}^t, \mathbf{m}^t) + \tilde{\mathbf{m}}^t \cdot \dot{\mathbf{m}}^t]\, dt - \ln\langle 0|S|0\rangle \qquad (45)$$

The second term of (45), which still contains $\tilde{\mathbf{h}}^t$ and $\mathbf{h}^t$,must now be expressed entirely in terms of $\tilde{\mathbf{m}}^t$ and $\mathbf{m}^t$ through (42). The change of variables from $\tilde{\mathbf{h}}^t$ and $\mathbf{h}^t$ to $\tilde{\mathbf{m}}^t$ and $\mathbf{m}^t$ associated with the Legendre transformation is the analog of the corresponding well-known change of variables from the fugacity to the density in the theory of classical fluids. Once we choose $\tilde{\mathbf{m}}^t$ and $\mathbf{m}^t$ to be independent variables, the conditions (44) guarantee the absence of articulation lines from the graphs of $\langle 0|S|0\rangle$, which hence eliminates long-range correlations contained in $W\{\tilde{\mathbf{h}}, \mathbf{h}\}$.[3] A few typical diagrams of $\Gamma\{\tilde{\mathbf{m}}, \mathbf{m}\}$ are shown in Fig. 5.

From (42) and (45) we find

$$\tilde{\mathbf{h}}^t = \delta\Gamma\{\tilde{\mathbf{m}}, \mathbf{m}\}/\delta\tilde{\mathbf{m}}^t \qquad (46a)$$

$$\mathbf{h}^t = \delta\Gamma\{\tilde{\mathbf{m}}, \mathbf{m}\}/\delta\mathbf{m}^t \qquad (46b)$$

These equations turn out to be the equations of motion for $\tilde{\mathbf{m}}^t$ and $\mathbf{m}^t$ for given $\{\tilde{\mathbf{h}}^t\}$ and $\{\mathbf{h}^t\}$. Now, the source fields are introduced here as artificial, unphysical fields, and hence are set to be zero in (46a) and (46b). Furthermore,

[3] It is true that the conditions (42a) and (42b) imply the absence of articulation lines also in W. However, the differentiations of W with respect to $\tilde{\mathbf{h}}^t$ and $\mathbf{h}^t$ to obtain response and correlation functions are to be taken at fixed $\{\tilde{\mathbf{m}}^t\}$ and $\{\mathbf{m}^t\}$, where the conditions (42a) and (42b) will have to be violated.

Fig. 5. Diagrams of $\Gamma\{\tilde{\mathbf{m}}, \mathbf{m}\}$.

$\tilde{\mathbf{m}}^t$, which is the average of $\tilde{\mathbf{S}}^t$, is also zero in any physical state. On the other hand, the stochastic operator $H(\tilde{\mathbf{S}}, \mathbf{S})$ has a general structure

$$\sum_i \tilde{\mathbf{S}}_i \mathscr{H}_i(\tilde{\mathbf{S}}, \mathbf{S}) \tag{47}$$

Hence $H(\tilde{\mathbf{m}}^t = 0, \mathbf{m}^t) = 0$ and $l \geqslant 1$ in (6). Thus $\ln\langle 0|S|0\rangle$ consists of graphs with free ends to the left, excluding those, such as the one shown in Fig. 3a, that are not allowed. Therefore we conclude by (45) that

$$\Gamma\{\tilde{\mathbf{m}} = 0, \mathbf{m}\} = 0 \tag{48}$$

implying that both sides of (46b) vanish in this case. The equation of motion for $\mathbf{m}^t$ is thus given by

$$\left[\frac{\delta\Gamma\{\tilde{\mathbf{m}}, \mathbf{m}\}}{\delta\tilde{\mathbf{m}}^t}\right]_{\{\tilde{\mathbf{m}}\}=\{0\}} = 0 \tag{49}$$

or, by (45), by the following:

$$\dot{\mathbf{m}}^t = -\left[\frac{\partial H(\tilde{\mathbf{m}}^t, \mathbf{m}^t)}{\partial\tilde{\mathbf{m}}^t}\right]_{\tilde{\mathbf{m}}^t=0} + \left[\frac{\delta \ln\langle 0|\hat{S}|0\rangle}{\delta\tilde{\mathbf{m}}^t}\right]^{\mathrm{OPI}}_{\{\tilde{\mathbf{m}}\}=\{0\}} \tag{50}$$

where we have modified the last term of (50) as follows. First, the absence of articulation lines enables us to replace $H'(t)$ by $\hat{H}'(t)$, given by

$$\hat{H}'(t) \equiv \frac{1}{2}\sum_{ij} J^t_{02}(ij)\phi_i(t)\phi_j(t) + \sum_{ln}{}' \frac{1}{l!\,n!}\sum_{\{i\}}\sum_{\{j\}} J^t_{ln}(\{i\}\{j\})\tilde{\phi}_{i_1}(t)\cdots\tilde{\phi}_{i_l}(t)\phi_{j_1}(t)\cdots\phi_{j_n}(t) \tag{51}$$

where $\sum'_{ln}$ is the sum over l and n such that $l + n \geqslant 3$. With this, $\hat{S}$ is

$$\hat{S} \equiv \exp_+\left[-\int_{t_0}^{t_f} dt\, \hat{H}'(t)\right] \tag{52}$$

The superscript OPI (one-particle irreducible) means that all the diagrams with articulation lines from the resulting perturbation expansion have to be removed.

4. TDGL MODEL

Here we illustrate our general method by deriving the macroscopic equation of motion for m^t of the TDGL model near the critical point. The model is given by

$$H(\tilde{\mathbf{S}}, \mathbf{S}) = -\int d\mathbf{r}\, \tilde{S}(\mathbf{r})L[\tilde{S}(\mathbf{r}) - (\tau - \nabla^2)S(\mathbf{r}) - \tfrac{1}{6}gS(\mathbf{r})^3] \tag{53}$$

In order to eliminate divergences at short wavelength, we renormalize as follows:

$$\begin{aligned}&S \to Z^{1/2}S, \qquad \tilde{S} \to Z^{-1/2}\tilde{S}, \qquad L \to Z_L^{-1}L \\ &\tau \to (Z_2\tau + \delta\tau)/Z, \qquad g \to Z_1 Z^{-2} g \\ &t \to (Z/L)t, \qquad H \to (L/ZZ_L)(H + H_{CT}), \qquad \Gamma \to Z_L^{-1}\Gamma\end{aligned} \tag{54}$$

where $\delta\tau$ and the Z's are the renormalization constants, which contain all the divergences as the upper cutoff wave number goes to infinity. We then obtain

$$H(\tilde{\mathbf{S}}, \mathbf{S}) = -\int d\mathbf{r}\, \tilde{S}(\mathbf{r})[\dot{S}(\mathbf{r}) - (\tau - \nabla^2)S(\mathbf{r}) - \tfrac{1}{6}gS(\mathbf{r})^3] \tag{55}$$

The counter term H_{CT} containing Z, Z_1, Z_2, and $\delta\tau$ is given by

$$H_{CT} = \int d\mathbf{r}\, \tilde{S}\{[(Z_2 - 1)\tau + \delta\tau - (Z - 1)\nabla^2]S + \tfrac{1}{6}g(Z_1 - 1)S^3\} \tag{56}$$

The renormalization (54) was chosen so as to separate the renormalization associated with a purely dynamic quantity like L from that of the static part. Then,

$$\Gamma = \int dx\, [H(\tilde{m}^t, m^t) + H_{CT}(\tilde{m}^t, m^t)] + Z_L \int \tilde{m}^t \dot{m}^t\, dx - Z_L[\ln\langle 0|\hat{S}|0\rangle]^{\mathrm{OPI}} \tag{57}$$

where

$$\hat{S} = \exp_+\left\{-Z_L^{-1}\int dt\, [\hat{H}'(t) + H'_{CT}(t)]\right\} \tag{58}$$

$\hat{H}'(t)$ is related to $\hat{H}'^t$, which is the part of H that corresponds to (51), by

$$\hat{H}'(t) = \left[\exp_+\left(-Z_L^{-1}\int_t^{t_f} ds\, H_g^s\right)\right]\hat{H}'^t \exp_-\left(Z_L^{-1}\int_t^{t_f} ds\, H_g^s\right) \tag{59}$$

where H_g^t is the part of H that corresponds to (22). Explicitly, we have

$$\begin{aligned}H_g^t = \int \{&-\dot{m}^t\tilde{\phi}^t + \dot{\tilde{m}}^t\phi^t - \dot{m}^t\tilde{m}^t - (\tilde{\phi}^t)^2 \\ &+ \tilde{\phi}^t[(\tau - \nabla^2)\phi^t + \tfrac{1}{2}g(m^t)^2]\phi^t\}\, dr\end{aligned} \tag{60}$$

$$\hat{H}'^t = \tfrac{1}{2}g\int [m^t\tilde{m}^t(\phi^t)^2 + m^t\tilde{\phi}^t(\phi^t)^2 + \tfrac{1}{3}\tilde{m}^t(\phi^t)^3 + \tfrac{1}{3}\tilde{\phi}^t(\phi^t)^3]\, dr \tag{61}$$

$H'_{CT}(t)$ is related to H'^t_{CT} as in (59), where

$$H'^t_{CT} \equiv H_{CT}(\tilde{m}^t + \tilde{\phi}^t, m^t + \phi^t) - H_{CT}(\tilde{m}^t, m^t) \tag{62}$$

We now derive the macroscopic equation of motion for m^t that corresponds to (49) when the dimensionality of space d is $4 - \epsilon$ with small ϵ. The re-

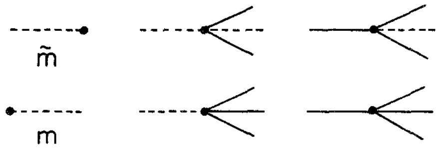

Fig. 6. Vertices for the TDGL model.

normalization group theory of critical phenomena tells us that the renormalized coupling constant g is of the order of ϵ. Hence the last term of (57) can be treated by a perturbation theory, where, however, we suppose that gm^2 is not small, as it is in the ordered phase. The building blocks of the diagram then consist of the two types of propagators G and F, as shown in Fig. 1 and the vertices as shown in Fig. 6 and those for $H'_{CT}(t)$, which are not shown explicitly. The propagators G and F now satisfy the following equations corresponding to (39) and (40):

$$\frac{\partial}{\partial t} G(\mathbf{x}\mathbf{x}') = -\frac{1}{Z_L}\left[\tau - \nabla^2 + \frac{g}{2} m(\mathbf{x})^2\right] G(\mathbf{x}\mathbf{x}') + \delta(\mathbf{x} - \mathbf{x}') \tag{63}$$

$$\frac{\partial}{\partial t} F(\mathbf{x}\mathbf{x}') = -\frac{1}{Z_L}\left[\tau - \nabla^2 + \frac{g}{2} m(\mathbf{x})^2\right] F(\mathbf{x}\mathbf{x}') + \frac{2}{Z_L} G(\mathbf{x}'\mathbf{x}) \tag{64}$$

where

$$\mathbf{x} = (\mathbf{r}, t)$$

These propagators incorporate the effect of nonlinear coupling arising from averaged variable $m(\mathbf{x})$, and hence we call them the mean field propagators.

The terms up to $O(g^2)$ of $[\ln\langle 0|\hat{S}|0\rangle]^{\text{OPI}}$ are shown graphically in Fig. 7, where the cross denotes $[\delta\tau + (Z_2 - 1)\tau - (Z - 1)\nabla^2]Z_L\, \delta(\mathbf{x} - \mathbf{x}')$, which arises from the counter term.

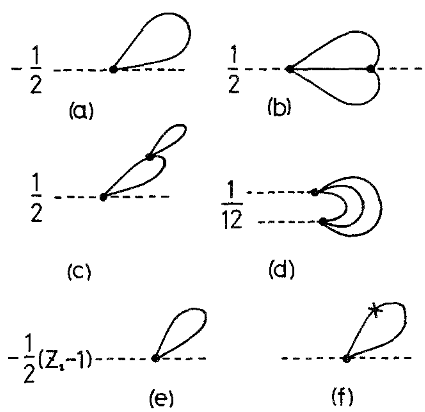

Fig. 7. Diagrams for $\Gamma\{\tilde{\mathbf{m}}, \mathbf{m}\}$ of the TDGL model.

The constants $\delta\tau$ and the Z's are determined by the normalization conditions. For this purpose we introduce the following functions:

$$\hat{\Gamma}^{(l,n)}(\{\mathbf{q}\}, \tau) = \int_{-\infty}^{\infty} dt_2 \cdots \int_{-\infty}^{\infty} dt_{l+n} \times \left[\frac{\delta^{l+n}\Gamma}{\delta\tilde{m}_{\mathbf{q}_1}(t_1)\cdots\delta\tilde{m}_{\mathbf{q}_l}(t_l)\,\delta m_{\mathbf{q}_{l+1}}(t_{l+1})\cdots\delta m_{\mathbf{q}_{l+n}}(t_{l+n})}\right]_{\tilde{\mathbf{m}}=0,\mathbf{m}=0} \tag{65}$$

where we have introduced Fourier components

$$m_{\mathbf{q}}(t) = \int m(\mathbf{x}) \exp(-i\mathbf{qr})\, d\mathbf{r}$$
$$\tilde{m}_{\mathbf{q}}(t) = \int \tilde{m}(\mathbf{x}) \exp(i\mathbf{qr})\, d\mathbf{r} \tag{66}$$

The five normalization conditions for determining the five constants $\delta\tau$, Z, Z_1, Z_2, and Z_L are chosen to be

$$\hat{\Gamma}^{(1,1)}(\mathbf{q} = 0, \tau = 0) = 0 \tag{67a}$$

$$\left.\frac{\partial\hat{\Gamma}^{(1,1)}}{\partial\tau}\right|_{\mathbf{q}=0,\tau=\mu^2} = 1 \tag{67b}$$

$$\left.\frac{\partial\hat{\Gamma}^{(1,1)}}{\partial(q^2)}\right|_{\mathbf{q}=0,\tau=\mu^2} = 1 \tag{67c}$$

$$\hat{\Gamma}^{(1,3)}(q_i = 0, \tau = \mu^2) = g = u\mu^{\epsilon} \tag{67d}$$

$$\hat{\Gamma}^{(2,0)}(q_i = 0, \tau = \mu^2) = -2Z_L \tag{67e}$$

where μ is the reference wave number of normalization and u is the dimensionless coupling constant. Note that (67a)–(67c) and (67e) are consistent with the fluctuation-dissipation theorem.[4] The results for $\delta\tau$, Z, Z_1, and Z_2 reduce to those already known, and we only consider Z_L. Namely, we have with $\int_{\mathbf{k}} \equiv (2\pi)^{-d}\int d\mathbf{k}$

$$-2Z_L = \hat{\Gamma}^{(2,0)}(q_i = 0, \tau = \mu^2)$$
$$= -2 + \frac{g^2}{6Z_L}\int_{-\infty}^{\infty} dt \int_{\mathbf{k}}\int_{\mathbf{p}} F_{\mathbf{k}}^0(t)F_{\mathbf{p}}^0(t)F^0_{-\mathbf{k}-\mathbf{p}}(t) \tag{68}$$

where $F_{\mathbf{k}}^0(t)$ is the mean field propagator with $\mathbf{m} = 0$, given by

$$F_{\mathbf{k}}^0(t) = \{\exp[-(\gamma_k^0/Z_L)|t|]\}/\gamma_{\mathbf{k}}^0 \tag{69}$$

[4] In terms of the functions Γ_{nl} introduced in Ref. 4 this theorem is

$$-(i\omega/Z_L)\Gamma_{20}(\mathbf{q}\omega) = \Gamma_{11}(\mathbf{q}\omega) - \Gamma_{11}(-\mathbf{q} - \omega)$$

with

$$\gamma_{\mathbf{k}}^{0} \equiv \mu^2 + k^2 \tag{70}$$

That is,

$$Z_L = 1 - \frac{g^2}{6}\int_{\mathbf{k}}\int_{\mathbf{p}} \frac{1}{\gamma_{\mathbf{k}}^{0}\gamma_{\mathbf{p}}^{0}\gamma_{\mathbf{k}+\mathbf{p}}^{0}(\gamma_{\mathbf{k}}^{0} + \gamma_{\mathbf{p}}^{0} + \gamma_{\mathbf{k}+\mathbf{p}}^{0})} \tag{71}$$

We evaluate the integration in $(4 - \epsilon)$ dimensions to obtain

$$Z_L = 1 - [(Su)^2/\epsilon]\tfrac{1}{8}\ln\tfrac{4}{3} \tag{72}$$

where

$$S = \frac{2\pi^{d/2}}{\Gamma(d/2)}\frac{1}{(2\pi)^d}$$

Now we want to derive the renormalized macroscopic equation of motion for $m(t) \equiv m_{\mathbf{q}=0}(t)$, which can be separated into two parts, i.e., the instantaneous part and the remainder, which may be called the noninstantaneous part.[7] The instantaneous part describes the time evolution of m^t if the system is always in local equilibrium with given m^t. It is expressed in terms of the equation of state. By a perturbation calculation in g the instantaneous part and the remainder are calculated up to the order g and up to the order g^2, respectively.

First let us write the Fourier-transformed propagators G and F, which obey (62) and (63), as

$$G_{\mathbf{k}}(t_1, t_2) = \exp\left[-\int_{t_2}^{t_1}\gamma_{\mathbf{k}}(s)\,ds/Z_L\right] \tag{73}$$

$$F_{\mathbf{k}}(t_1, t_2) = \left\{\exp\left[-\int_{t_2}^{t_1}\gamma_{\mathbf{k}}(s)\,ds/Z_L\right]\right\}I_{\mathbf{k}}(t_2) \tag{74}$$

for $t_1 > t_2$, where

$$\gamma_{\mathbf{k}}(t) = k^2 + \tau + \tfrac{1}{2}gm^2(t) \tag{75}$$

and the variance $I_{\mathbf{k}}(t) = F_{\mathbf{k}}(tt)$ obeys the following equation:

$$Z_L\,(\partial/\partial t)I_{\mathbf{k}}(t) = 2 - 2\gamma_{\mathbf{k}}(t)I_{\mathbf{k}}(t) \tag{76}$$

With the help of (76) we can separate out the instantaneous part of the following macroscopic equation of motion obtained from (57):

$$Z_L\dot{m}(t) = -\left\{\tau m + \frac{g}{6}m^3 + [(Z_2 - 1)\tau + \delta\tau]m + \frac{g}{6}(Z_1 - 1)m^3\right\}$$
$$+ Z_L\frac{\partial}{\partial\tilde{m}(t)}[\ln\langle 0|\hat{S}|0\rangle]^{\mathrm{OPI}}_{\{\tilde{m}\}=0} \tag{77}$$

The last term is obtained from $[\ln\langle 0|\hat{S}|0\rangle]^{\rm OPI} = (a) + (b) + (c) \cdots$; these terms are given graphically in Fig. 7. Explicitly, we have

$$Z_L \frac{\delta}{\delta\tilde{m}(t)}(a)\Big|_{\{\tilde{m}\}=0} = -\frac{g}{2}m(t)\int_{\mathbf{b}} F_{\mathbf{q}}(t,t)$$
$$= -\frac{g}{2}m(t)\left[\int_{\mathbf{q}}\frac{1}{\gamma_{\mathbf{q}}(t)} - \int_{\mathbf{q}}\frac{Z_L}{2\gamma_{\mathbf{q}}(t)}\dot{I}_{\mathbf{q}}(t)\right] \tag{78}$$

where we have used (74) and (76). The first term of (78) is the instantaneous part, which diverges at short wavelengths. This divergence is canceled by the counter term in (77). The second term, which contributes to the noninstantaneous part, contains no divergence and is obtained as

$$\frac{Sgm(t)Z_L^{(d/2)-1}}{2^{(d/2)+1}}\Gamma\left(\frac{d}{2}\right)\int_0^\infty ds\, s^{-d/2}\left\{\exp\left[-\frac{2}{Z_L}\gamma_0(t)s\right]\right.$$
$$\left.-\exp\left[-\frac{2}{Z_L}\int_0^s ds'\,\gamma_0(t-s')\right]\right\} \tag{79}$$

where use has been made of the equation

$$I_{\mathbf{q}}(t) = \frac{2}{Z_L}\int_0^\infty ds\,\exp\left[-\frac{2}{Z_L}\int_0^s ds'\,\gamma_{\mathbf{q}}(t-s')\right] \tag{80}$$

which is obtained by integrating (79) with the boundary condition $I_{\mathbf{q}}(-\infty) = 0$. The contributions from the terms (b) and (c) are more complicated. However, as is shown in the appendix, we can extract the ultraviolet divergences as well as the contributions to the instantaneous part. The evaluation of these contributions becomes so complicated that here we will be content with the contribution linear in m, which is obtained from the second term of (A.6) by setting all the m in G and I^0 equal to zero. The ultraviolet divergence of the noninstantaneous part in (A.6) is contained in this linear term. By making use of the normalization conditions (67), we eliminate these divergences and we obtain the following scaled equation of motion with $\mu = 1$:

$$\frac{\partial\hat{m}}{\partial\hat{t}} = -F(\hat{m}) + \frac{(Su^*)\Gamma(2-\frac{1}{2}\epsilon)}{2^{3-(\epsilon/2)}}\hat{m}(\hat{t})\int_0^\infty d\hat{s}\,\hat{s}^{-2+(\epsilon/2)}\left\{\exp[-2\hat{\gamma}_0(\hat{t})\hat{s}]\right.$$
$$\left.-\exp\left[-2\int_0^{\hat{s}} d\hat{s}'\,\hat{\gamma}_0(\hat{t}-\hat{s}')\right]\right\}$$
$$+\frac{(Su^*)^2}{6}\int_0^\infty d\hat{s}\,\sigma(\hat{s})\left[\frac{\partial\hat{m}(\hat{t})}{\partial\hat{t}} - \frac{\partial\hat{m}(\hat{t}-\hat{s})}{\partial\hat{t}}\right] \tag{81}$$

with $\hat{\gamma}_0(\hat{t}) \equiv 1 + \frac{1}{2}u^*\hat{m}^2(\hat{t})$,

$$\sigma(\hat{t}) = \frac{1}{4\hat{s}}\int_1^\infty ds_1 \int_1^\infty ds_2 \int_1^\infty ds_3 \frac{\exp[-\hat{s}(s_1 + s_2 + s_3)]}{[s_1 s_2 + (s_1 + s_2)s_3]^2} \tag{82}$$

and

$$F(\hat{m}) = \hat{m}\left\{1 + \frac{u^*}{6}\hat{m}^2 + \frac{\epsilon}{6}\left(1 + \frac{u^*}{2}\hat{m}^2\right)\left[\ln\left(1 + \frac{u^*}{2}\hat{m}^2\right) - 1\right]\right\} + O(\epsilon^2) \tag{83}$$

where we have introduced the scaling variables $\hat{t} = \tau^{z\nu}t$ and $\hat{m} = m\tau^{-\beta}$, and the fixed-point value u^* of u through $Su^* = (2\epsilon/3)(1 + 7\epsilon/54 + \cdots)$. The meanings and the values of the exponents are the same as those found in the literature.[1–8] Strictly speaking, (81) is only correct to the order ϵ. However, the present simple example shows how the renormalization of the higher order terms can be carried out. Equation (81) coincides with the result obtained recently by Bausch and Janssen[8] to the first order in ϵ.

5. CONCLUDING REMARKS

In the preceding sections we have developed a method for finding the generating functional Γ and have derived from it the macroscopic evolution equation for the average order parameter. The method closely parallels that of equilibrium critical phenomena,[1] and the various results in equilibrium critical phenomena can be readily extended to our problem. We have already derived the dynamical scaling extended to this problem.[5] The analyticity requirements for the equation of state would imply similar requirements for the macroscopic evolution equation.

APPENDIX

The contributions to the macroscopic equation of motion from Fig. 7 will be considered. The following formulas are useful for this purpose [see (76)]:

$$I_{\mathbf{k}}(t) \equiv F_{\mathbf{k}}(tt) = I_{\mathbf{k}}^0(t) + I_{\mathbf{k}}'(t) \tag{A.1}$$

with

$$I_{\mathbf{k}}^0(t) = [\gamma_{\mathbf{k}}(t)]^{-1}, \qquad I_{\mathbf{k}}'(t) = -Z_L\dot{I}_{\mathbf{k}}(t)I_{\mathbf{k}}^0(t)/2$$

For $t > s$ we have

$$F_{\mathbf{k}}(ts) = G_{\mathbf{k}}(ts)I_{\mathbf{k}}(s) \tag{A.2}$$

and by (64)

$$G_{\mathbf{k}}(ts) = Z_L \frac{\partial F_{\mathbf{k}}(ts)}{\partial s} + G_{\mathbf{k}}'(ts) \tag{A.3}$$

with

$$G_{\mathbf{k}}'(ts) = -Z_L \dot{I}_{\mathbf{k}}(s) G_{\mathbf{k}}(ts)/2 \tag{A.4}$$

The term (a) in Fig. 7 was considered in Section 4. The term (b) gives

$$\begin{aligned} Z_L \frac{\delta}{\delta \tilde{m}(t)} (\mathrm{b}) &= \frac{g^2}{2Z_L} \int_{\mathbf{k}} \int_{\mathbf{p}} \int_{-\infty}^{t} ds\, F_{\mathbf{k}}(ts) F_{\mathbf{p}}(ts) G_{-\mathbf{k}-\mathbf{p}}(ts) m(s) \\ &= \frac{g^2}{2} \int_{\mathbf{k}} \int_{\mathbf{p}} \int_{-\infty}^{t} ds\, F_{\mathbf{k}}(ts) F_{\mathbf{p}}(ts) \frac{\partial F_{-\mathbf{k}-\mathbf{p}}(ts)}{\partial s} m(s) \\ &\quad + \frac{g^2}{2Z_L} \int_{\mathbf{k}} \int_{\mathbf{p}} \int_{-\infty}^{t} ds\, F_{\mathbf{k}}(ts) F_{\mathbf{p}}(ts) G_{\mathbf{k}}'(ts) m(s) \end{aligned} \tag{A.5}$$

The second term has no ultraviolet divergence in its momentum integrals. By making use of Eqs. (A.2) and (A.3), the first term is further reduced to

$$\begin{aligned} &\tfrac{1}{6} g^2 \int_{\mathbf{k}} \int_{\mathbf{p}} \int_{-\infty}^{t} ds\, (\partial/\partial s)[F_{\mathbf{k}}(ts) F_{\mathbf{p}}(ts) F_{-\mathbf{k}-\mathbf{p}}(ts)] m(s) \\ &= \tfrac{1}{6} g^2 \int_{\mathbf{k}} \int_{\mathbf{p}} I_{\mathbf{k}}(t) I_{\mathbf{p}}(t) I_{-\mathbf{k}-\mathbf{p}}(t) m(t) \\ &\quad - \tfrac{1}{6} g^2 \int_{\mathbf{k}} \int_{\mathbf{p}} \int_{-\infty}^{t} ds\, F_{\mathbf{k}}(ts) F_{\mathbf{p}}(ts) F_{-\mathbf{k}-\mathbf{p}}(ts) \dot{m}(s) \\ &= \tfrac{1}{6} g^2 \int_{\mathbf{k}} \int_{\mathbf{p}} I_{\mathbf{k}}^0(t) I_{\mathbf{p}}^0(t) I_{-\mathbf{k}-\mathbf{p}}^0(t) m(t) \\ &\quad - \tfrac{1}{6} g^2 \int_{\mathbf{k}} \int_{\mathbf{p}} \int_{-\infty}^{t} ds\, G_{\mathbf{k}}(ts) G_{\mathbf{p}}(ts) G_{-\mathbf{k}-\mathbf{p}}(ts) I_{\mathbf{k}}^0(s) I_{\mathbf{p}}^0(s) I_{-\mathbf{k}-\mathbf{p}}^0(s) \dot{m}(s) \\ &\quad + \tfrac{1}{6} g^2 \Big[\int_{\mathbf{k}} \int_{\mathbf{p}} J_{\mathbf{k},\mathbf{p},-\mathbf{k}-\mathbf{p}}(t) m(t) \\ &\quad - \int_{\mathbf{k}} \int_{\mathbf{p}} \int_{-\infty}^{t} ds\, G_{\mathbf{k}}(ts) G_{\mathbf{p}}(ts) G_{-\mathbf{k}-\mathbf{p}}(ts) J_{\mathbf{k},\mathbf{p},-\mathbf{k}-\mathbf{p}}(s) \dot{m}(s) \Big] \end{aligned} \tag{A.6}$$

with

$$J_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}(t) \equiv I_{\mathbf{k}_1}(t) I_{\mathbf{k}_2}(t) I_{\mathbf{k}_3}(t) - I_{\mathbf{k}_1}^0(t) I_{\mathbf{k}_2}^0(t) I_{\mathbf{k}_3}^0(t) \tag{A.7}$$

The third term has no ultraviolet divergence in its momentum integrals. The expression for the term (c) in Fig. 7 is

$$Z_L \frac{\delta(\mathrm{c})}{\delta\tilde{m}(t)} = \frac{g^2 m(t)}{2Z_L} \int_{\mathbf{k}} \int_{\mathbf{p}} \int_{-\infty}^{t} ds\, G_{\mathbf{k}}(ts) F_{-\mathbf{k}}(ts) I_{\mathbf{p}}(s)$$
$$= \frac{g^2}{2} m(t) \int_{\mathbf{k}} \int_{\mathbf{p}} \int_{-\infty}^{t} ds \frac{\partial F_{\mathbf{k}}(ts)}{\partial s} F_{-\mathbf{k}}(ts) I_{\mathbf{p}}(s)$$
$$+ \frac{g^2}{2Z_L} m(t) \int_{\mathbf{k}} \int_{\mathbf{p}} \int_{-\infty}^{t} ds\, G_{\mathbf{k}}'(ts) F_{-\mathbf{k}}(ts) I_{\mathbf{p}}(s) \qquad \text{(A.8)}$$

The ultraviolet divergence in the p integral in (A.8) is canceled by its counter term (f) in Fig. 7. The second term gives no ultraviolet divergence in its k integral. The first term becomes, on using Eqs. (A.2) and (A.3),

$$\tfrac{1}{4} g^2 m(t) \int_{\mathbf{k}} \int_{\mathbf{p}} \int_{-\infty}^{t} ds\, [\partial F_{\mathbf{k}}(ts) F_{-\mathbf{k}}(ts)/\partial s] I_{\mathbf{p}}(s)$$
$$= \tfrac{1}{4} g^2 m(t) \int_{\mathbf{k}} \int_{\mathbf{p}} I_{\mathbf{k}}(t) I_{-\mathbf{k}}(t) I_{\mathbf{p}}(t)$$
$$- \tfrac{1}{4} g^2 m(t) \int_{\mathbf{k}} \int_{\mathbf{p}} \int_{-\infty}^{t} ds\, F_{\mathbf{k}}(ts) F_{-\mathbf{k}}(ts) \dot{I}_{\mathbf{p}}(s)$$
$$= \tfrac{1}{4} g^2 m(t) \int_{\mathbf{k}} \int_{\mathbf{p}} I_{\mathbf{k}}^0 I_{-\mathbf{k}}^0(t) I_{\mathbf{p}}^0(t) + \tfrac{1}{4} g^2 m(t) \int_{\mathbf{k}} \int_{\mathbf{p}} J_{\mathbf{k},-\mathbf{k},\mathbf{p}}(t)$$
$$- \tfrac{1}{4} g^2 m(t) \int_{\mathbf{k}} \int_{\mathbf{p}} \int_{-\infty}^{t} ds\, F_{\mathbf{k}}(ts) F_{-\mathbf{k}}(ts) \dot{I}_{\mathbf{p}}(s) \qquad \text{(A.9)}$$

Therefore, the ultraviolet divergence only remains in the k integral of the first term of (A.9) after adding the contribution from (f), which can be simply obtained by replacing $\int_{\mathbf{p}} I_{\mathbf{p}}(s)$ in Eq. (A.8) by $-[\delta\tau + (Z_2 - 1)\tau + (Z - 1)k^2]$. The resulting ultraviolet divergences, including the terms in Fig. 7, are canceled by the terms H_{CT} and the second term of (57) as illustrated in the text (Section 4). Since the terms in (A.5) and (A.8) that have no ultraviolet divergence also contribute to the final form of the equation of motion, the scaled equation of motion becomes so complicated that the full expression will not be given here. Instead, we will be content with the simple approximate expression (81).

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